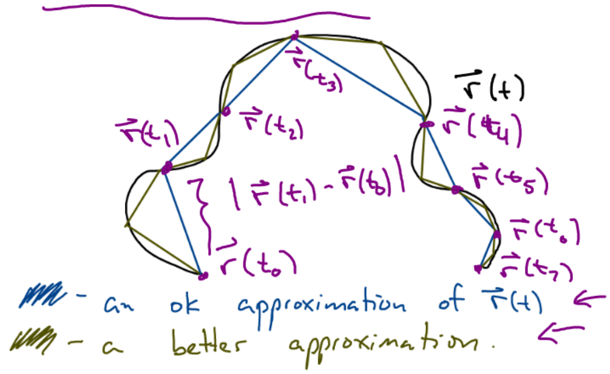
Arc Length

Let $\vec{r}(t)$ be a (parametrization of a) space curve. The total distance travelled by $\vec{r}(t)$ over the interval $a \le t \le b$ is obtained by limiting some piecewise-linear approximations of the curve to the curve itself, by partitioning the interval into progressively finer subintervals.



The approximation corresponding to partition $[a = t_0 < t_1 < \cdots < t_n = b]$ is given precisely by $S_n = \sum_{k=1}^{n} |\vec{r}(t_k) - \vec{r}(t_{k-1})|$. If we further impose $t_k = t_{k-1} + \Delta_n t$ (where $\Delta_n t = \frac{b-a}{n}$), then we seek to compute

$$S = \lim_{n\to\infty} S_n$$

$\Delta_n t \to 0$ as $n \to \infty$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} |\vec{r}(a + k\Delta_n t) - \vec{r}(a + (k-1)\Delta_n t)| \frac{\Delta_n t}{\Delta_n t}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left| \frac{\vec{r}(a + k\Delta_n t) - \vec{r}(a + (k-1)\Delta_n t)}{\Delta_n t} \right| \Delta_n t$$

$$= \int_{t=a}^{b} |\vec{r}\,'(t)|\, dt$$

We will take this as a definition! In particular, when $\vec{r}(t)$ traverses every point of its curve once we obtain:

Same formula

Def^n: The arc length of curve $\vec{r}(t)$ from $t=a$ to $t=b$ is

$$s = \int_{t=a}^{b} |\vec{r}\,'(t)|\, dt$$

Ex: Compute the length of the curve $\vec{r}(t)$.

ⓐ $\vec{r}(t) = \langle t, 3\cos(t), 3\sin(t)\rangle$ on $-5 \le t \le 5$.

Sol: $s = \int_{t=a}^{b} |\vec{r}\,'(t)|\,dt$

$a = -5,\ b = 5$

$\vec{r}\,'(t) = \langle 1, -3\sin(t), 3\cos(t)\rangle$

$|\vec{r}\,'(t)| = \sqrt{1^2 + 3^2\sin^2(t) + 3^2\cos^2(t)}$

$= \sqrt{1+9} = \sqrt{10}$

$= \int_{t=-5}^{5} \sqrt{10}\,dt$

$= \left[\sqrt{10}\,t\right]_{t=-5}^{5} = \sqrt{10}\,(5-(-5)) = 10\sqrt{10}$ ☑

ⓑ $\vec{r}(t) = 2t\,\vec{i} + t^2\,\vec{j} + \frac{1}{3}t^3\,\vec{k}$ on $0 \le t \le 1$.

Sol: $s = \int_{t=a}^{b} |\vec{r}\,'(t)|\,dt$

$a = 0,\ b = 1$

$\vec{r}(t) = \langle 2t, t^2, \frac{1}{3}t^3\rangle$

$\vec{r}\,'(t) = \langle 2, 2t, t^2\rangle$

$|\vec{r}\,'(t)| = \sqrt{2^2 + 2^2t^2 + t^4}$

$= \sqrt{4 + 4t^2 + (t^2)^2}$

$= \sqrt{(2+t^2)^2} = 2 + t^2$

$= \int_{t=0}^{1} (2+t^2)\,dt$

$= \left[2t + \frac{1}{3}t^3\right]_{t=0}^{1}$

$= \left(2\cdot 1 + \frac{1}{3}\cdot 1\right) - \left(2\cdot 0 + \frac{1}{3}\cdot 0\right)$

$= 7/3 - 0 = 7/3$ ☑

ⓒ $\vec{r}(t) = \langle 1, t^2, t^3\rangle$ on $3 \le t \le 5$

Sol: $s = \int_{t=a}^{b} |\vec{r}\,'(t)|\,dt$

$a = 3,\ b = 5$

$\vec{r}\,'(t) = \langle 0, 2t, 3t^2\rangle$

$|\vec{r}\,'(t)| = \sqrt{0^2 + (2t)^2 + (3t^2)^2}$

$= \sqrt{t^2(4 + 9t^2)}$

$= t\sqrt{4+9t^2}$

↖ $t \ge 3 > 0$ ...

substitute → $= \int_{t=3}^{5} t\sqrt{4+9t^2}\,dt$

$\left(u = 4 + 9t^2\right.$

$du = 18t\,dt$

$u(3) = 85$

$u(5) = 229$

↑ double-check!

$= \int_{t=3}^{5} \frac{1}{18}\underbrace{\sqrt{4+9t^2}}_{u} \cdot \underbrace{18t\,dt}_{du}$

$= \int_{u=85}^{229} \frac{1}{18} u^{1/2}\,du$

$= \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_{u=85}^{229} = \frac{1}{27}\left(\sqrt{229}^{\,3} - \sqrt{85}^{\,3}\right)$ ☑

Arc length is fundamentally invariant of parametrization.
(i.e. if $\vec{\alpha}(t)$ and $\vec{\beta}(t)$ parametrize the same curve, then the arc length computation will yield the same result)

We like invariants like this because they give us freedom to choose how we compute.

As a nice application, we can use arc length to give curves a unit-speed parametrization! ($|\vec{r}'(t)| = 1$ for all t.) Let

$$s(t) := \int_{q=a}^{t} |\vec{r}'(q)|\, dq$$

be the arc length function.

In particular, $s'(t) = |\vec{r}'(t)|$ by the Fundamental Theorem of Calculus. If $|\vec{r}'(t)| \neq 0$ for all t (aka $\vec{r}(t)$ is a smooth parametrization), then $s(t)$ is a strictly increasing function. Hence s has an inverse function! As a result we can parametrize the curve by $\vec{p}(s) = \vec{r}(t(s))$, where $t(s)$ is the inverse of $s(t)$. Now note that

$$|\vec{p}\,'(s_0)| = |\vec{r}\,'(t(s_0))\, t'(s_0)|,$$

but by the inverse function theorem

$$t'(s_0) = \frac{1}{s'(t(s_0))} = \frac{1}{|\vec{r}\,'(t(s_0))|}.$$

Hence we obtain $|\vec{p}\,'(s_0)| = 1$.

Ex: Parametrize $\vec{r}(t) = \langle 5-t, 4t-3, 3t \rangle$ via arc length measured from the point $(4,1,3)$.

Sol: $\vec{r}\,'(t) = \langle -1, 4, 3 \rangle \quad \therefore \quad |\vec{r}\,'(t)| = \sqrt{1^2+4^2+3^2} = \sqrt{26}$

NB: $\vec{r}(a) = \langle 5-a, 4a-3, 3a \rangle = \langle 4,1,3 \rangle \implies a = 1$

$$\therefore \quad s(t) = \int_{q=a}^{t} |\vec{r}\,'(q)|\,dq = \int_{q=1}^{t} \sqrt{26}\,dq = \left[\sqrt{26}\,q\right]_{q=1}^{t} = \sqrt{26}\,(t-1).$$

$$s = \sqrt{26}\,(t-1) \iff \frac{s}{\sqrt{26}} = t-1 \iff t = 1 + \frac{1}{\sqrt{26}}\,s$$

Hence
$$\vec{p}(s) = \vec{r}(t(s)) = \vec{r}\left(1 + \frac{1}{\sqrt{26}}s\right).$$
$$= \left\langle 5 - \left(1 + \frac{1}{\sqrt{26}}s\right),\ 4\left(1+\frac{1}{\sqrt{26}}s\right) - 3,\ 3\left(1+\frac{1}{\sqrt{26}}s\right) \right\rangle$$
$$= \left\langle 4 - \frac{1}{\sqrt{26}}s,\ 1 + \frac{4}{\sqrt{26}}s,\ 3 + \frac{3}{\sqrt{26}}s \right\rangle \qquad ///$$

Check: $\vec{p}\,'(s) = \left\langle -\frac{1}{\sqrt{26}}, \frac{4}{\sqrt{26}}, \frac{3}{\sqrt{26}} \right\rangle$

$$\therefore \quad |\vec{p}\,'(s)| = \sqrt{\left(-\frac{1}{\sqrt{26}}\right)^2 + \left(\frac{4}{\sqrt{26}}\right)^2 + \left(\frac{3}{\sqrt{26}}\right)^2}$$
$$= \sqrt{\frac{1}{26} + \frac{16}{26} + \frac{9}{26}} = \sqrt{\frac{26}{26}} = 1$$

Ex: A particle with starting position $(0,0,3)$ moves 5 units along the curve $\vec{r}(t) = \langle 3\sin(t), 4t, 3\cos(t)\rangle$. Calculate its final position.

Sol: Start by parametrizing $\vec{r}(t)$ w.r.t. arc length.

$$s(t) = \int_{q=a}^{t} |\vec{r}\,'(q)|\,dq$$

$$\vec{r}(a) = \langle 0,0,3\rangle \iff a = 0$$

$$\vec{r}\,'(t) = \langle 3\cos(t), 4, -3\sin(t)\rangle$$

$$|\vec{r}\,'(t)| = \sqrt{3^2\cos^2(t) + 4^2 + 3^2\sin^2(t)} = \sqrt{3^2(\cos^2(t)+\sin^2(t)) + 4^2} = \sqrt{9\cdot 1 + 16} = \sqrt{25} = 5$$

$$= \int_{q=0}^{t} 5\,dq = \left[5q\right]_{q=0}^{t} = 5t$$

$\therefore$ $s = 5t \iff t = \frac{1}{5}s$ yields arc length parametrization

$$\vec{p}(s) = \vec{r}(t(s)) = \langle 3\sin(\tfrac{1}{5}s), \tfrac{4}{5}s, 3\cos(\tfrac{1}{5}s)\rangle$$

$\therefore$ the point 5 units along the curve is given by

$$\vec{p}(5) = \langle 3\sin(\tfrac{1}{5}\cdot 5), \tfrac{4}{5}\cdot 5, 3\cos(\tfrac{1}{5}\cdot 5)\rangle = \langle 3\sin(1), 4, 3\cos(1)\rangle.$$

//

Ex: Parametrize the curve $\vec{r}(t) = \langle \frac{2}{t^2+1} - 1, \frac{2t}{t^2+1} \rangle$ by arc length measured from point $(1,0)$.

use this...

Sol: $\vec{r}(a) = \langle 1, 0 \rangle \iff a = 0$ ← you check!

$$\vec{r}\,'(t) = \frac{d}{dt}\langle 2(t^2+1)^{-1} - 1, 2t(t^2+1)^{-1} \rangle$$

$$= \langle -2(t^2+1)^{-2} \cdot 2t, 2t \cdot -(t^2+1)^{-2} \cdot 2t + 2(t^2+1)^{-1} \rangle$$

$$= 2(t^2+1)^{-2} \langle -2t, -2t^2 + (t^2+1) \rangle$$

$$= -2(t^2+1)^{-2} \langle 2t, 1-t^2 \rangle$$

$$|\vec{r}\,'(t)| = |-2(t^2+1)^{-2}| \sqrt{(2t)^2 + (1-t^2)^2}$$

$$= 2(t^2+1)^{-2} \sqrt{4t^2 + 1 - 2t^2 + t^4}$$

$$= 2(t^2+1)^{-2} \sqrt{(t^2)^2 + 2t^2 + 1}$$

$$= 2(t^2+1)^{-2} \sqrt{(t^2+1)^2}$$

$$= 2(t^2+1)^{-2}(t^2+1) = 2(t^2+1)^{-1}$$

$$\therefore \quad s(t) = \int_{q=a}^{t} |\vec{r}\,'(q)|\, dq$$

$$= \int_{q=0}^{t} 2(q^2+1)^{-1}\, dq$$

$$= 2\left[\arctan(q)\right]_{q=0}^{t} = 2\arctan(t)$$

$$\therefore \quad s = 2\arctan(t) \iff t = \tan\left(\frac{s}{2}\right)$$

$$\therefore \quad \vec{p}(s) = \vec{r}(t(s)) = \left\langle \frac{2}{\tan^2(\frac{s}{2})+1} - 1, \frac{2\tan(\frac{s}{2})}{\tan^2(\frac{s}{2})+1} \right\rangle$$

$$= \left\langle 2\cos^2\left(\frac{s}{2}\right) - 1, 2\sin\left(\frac{s}{2}\right) \right\rangle$$

$\left(\text{for } -\frac{\pi}{2} < \frac{s}{2} < \frac{\pi}{2}\right)$ ////